ponding to $s_1^{l} s_2^{l}$ are identical term by term with the terms corresponding to $s_1 s_2$. Therefore the theorem follows that

$$D\Sigma\alpha_1^{p_1}\alpha_2^{p_2}\ldots\ldots\alpha_n^{p_n}=(p_1 p_2\ldots\ldots p_n)=\Sigma(p_{i_1}\ p_{i_2}+1\ p_{i_3}+2\ldots\ldots p_{i_n}+n-1),$$

where $i_1, i_2, \ldots\ldots i_n$ form a permutation of the numbers 1, 2, ... n.

B. Another Method. Elimination by Means of Symmetric Functions.

The problem of eliminating the variable between two binary forms by means of symmetric functions requires the calculation of the latter, and thus leads to the demand for symmetric functions as a whole. The calculation of all eliminants or resultants in succession is therefore, from this standpoint, the systematic calculation of all symmetric functions. In other words, the problems of calculating all resultants and of calculating all symmetric functions are identical. Given all resultants, we may write down the values of all symmetric functions. Given all symmetric functions, we may write down the values of all resultants. This idea is fruitful in giving rise to the following method of solving both problems simultaneously, and in yielding symmetric functions as a whole. The method will be illustrated and explained by one or two earlier cases, from which it will be seen that it can be carried as far as one pleases.

1. Two Quadratic Forms.

(1). *The Resultant.*

The resultant of two forms $a_0x^2+a_1x+a_2$ and $b_0x^2+b_1x+b_2$ of the second degree is (cf. p. 4)

$$b_0^2(a_0\beta_1^2+a_1\beta_1+a_2)(a_0\beta_2^2+a_1\beta_2+a_2)=$$

$$a_0^2(b_0\alpha_1^2+b_1\alpha_1+b_2)(b_0\alpha_2^2+b_1\alpha_2+b_2)=$$

$$b_0^2(a_0^2\Sigma\beta_1^2\beta_2^2+a_0a_1\Sigma\beta_1^2\beta_2+a_0a_2\Sigma\beta_1^2+a_1^2\Sigma\beta_1\beta_2+a_1a_2\Sigma\beta_1+a_2^2)=$$

$$a_0^2(b_0^2\Sigma\alpha_1^2\alpha_2^2+b_0b_1\Sigma\alpha_1^2\alpha_2+b_0b_2\Sigma\alpha_1^2+b_1^2\Sigma\alpha_1\alpha_2+b_1b_2\Sigma\alpha_1+b_2^2).$$

(2). *Aronhold's Operator.*

Applying Aronhold's Operator $\delta=b_0D_{a_0}+b_1D_{a_1}+b_2D_{a_2}$ on the first form of the resultant, first, using $b_0D_{a_0}$, then $b_1D_{a_1}$, and then $b_2D_{a_2}$, and denoting the coefficient of a_ia_k in the resultant by $|\,a_ia_k\,|=b_0^2\Sigma\beta_1^{2-i}\beta_2^{2-k}$, we get,

$$2a_0b_0\mid a_0^2\mid+b_0a_1\mid a_0a_1\mid+b_0a_2\mid a_0a_2\mid$$

$$+a_0b_1\mid a_0a_1\mid+2a_1b_1\mid a_1^2\mid+b_1a_2\mid a_1a_2\mid$$

$$+a_0b_2\mid a_0a_2\mid+a_1b_2\mid a_1a_2\mid+2a_2b_2\mid a_2^2\mid\equiv 0.$$

(3). *Identical Relations between Symmetric Functions.*

Since the expressions within the vertical strokes are functions of the b's,

and we can factor by columns, and take out the factors a_0, a_1, a_2 and since, for the rest, the whole expression is zero, whatever the values of the a's, it follows that their coefficients are zero, and identically zero, for they are zero for all values of the b's. We thus get :

$$2b_0 \mid a_0^2 \mid + b_1 \mid a_0a_1 \mid + b_2 \mid a_0a_2 \mid \equiv 0, \text{ coefficient of } a_0,$$

$$b_0 \mid a_0a_1 \mid + 2b_1 \mid a_1^2 \mid + b_2 \mid a_1a_2 \mid \equiv 0, \text{ coefficient of } a_1,$$

$$b_0 \mid a_0a_2 \mid + b_1 \mid a_1a_2 \mid + 2b_2 \mid a_2^2 \mid \equiv 0, \text{ coefficient of } a_2.$$

From these identities also follows :

$$\begin{vmatrix} 2 \mid a_0^2 \mid & \mid a_0a_1 \mid & \mid a_0a_2 \mid \\ \mid a_1a_0 \mid & 2 \mid a_1^2 \mid & \mid a_1a_2 \mid \\ \mid a_2a_0 \mid & \mid a_2a_1 \mid & 2 \mid a_2^2 \mid \end{vmatrix} \equiv 0,$$

an identical relation between the six symmetric functions of a quadratic form which enter into the resultant, with another quadratic form. It is seen to be a symmetric determinant.

(4). *Application of Relations to Calculate Symmetric Functions.*

We may use the preceding identities to find the symmetric functions involved, of which it may be taken for granted that we know

$$\mid a_0^3 \mid, \mid a_1^2 \mid, \mid a_2^2 \mid, \text{ and } \mid a_1a_2 \mid, \text{ or } b_0^2\Sigma(\beta_1\beta_2)^2, b_0^2\Sigma\beta_1\beta_2^2, b_0^2\Sigma(\beta_1\beta_2)^2,$$

$$\text{and } b_0^2\Sigma\beta_1 \text{ equal to } b_2^2,\ b_0b_2,\ b_0^2, \text{ and } -b_0b_1.$$

Using these values with the first two identities, we have :

$$b_1 \mid a_0a_1 \mid + b_2 \mid a_0a_2 \mid = -2b_0b_2^2$$

$$b_0 \mid a_0a_1 \mid + 0 \mid a_0a_2 \mid = -b_0b_1b_2.$$

Of these the second gives $\mid a_0a_1 \mid = -b_1b_2$, and by substituting this value in the first, $\mid a_0a_2 \mid = -2b_0b_2 + b_1^2$, the same results as appear in the table on page 5, when fx is changed into ϕx.

2. Two Cubic Forms.

We will next obtain relations between the symmetric functions which occur in the resultant of two forms of the third degree.

(1). *The Resultant.*

The resultant is equal to

$$a_0^3(b_0\alpha_1^3 + b_1\alpha_1^2 + b_2\alpha_1 + b_3)(b_0\alpha_2^3 + b_1\alpha_2^2 + b_2\alpha_2 + b_3)$$

$$\times (b_0\alpha_3^3 + b_1\alpha_3^2 + b_2\alpha_3 + b_3) = (-1)^{3\times 3}$$

$$b_0^3(a_0\beta_1^3+a_1\beta_1^2+a_2\beta_1+a_3)(a_0\beta_2^3+a_1\beta_2^2+a_2\beta_2+a_3)(a_0\beta_3^3+a_1\beta_3^2+a_2\beta_3+a_3)$$

$$=-(a_0^3\mid 0^3\mid +a_0^2a_1\mid 0^21\mid +a_0^2a_2\mid 0^22\mid +a_0^2a_3\mid 0^23\mid +a_0a_1^2\mid 01^2\mid$$

$$+a_0a_2^2\mid 02^2+a_0a_3^2\mid 03^2\mid +a_1^3\mid 1^3\mid +a_1^2a_2\mid 1^22\mid +a_1^2a_3\mid 1^23\mid$$

$$+a_1a_2^2\mid 12^2\mid +a_1a_3^2\mid 13^2\mid +a_2^3\mid 2^3\mid +a_2^2a_3\mid 2^23\mid +a_2a_3^2\mid 23^2\mid +a_3^3\mid$$

$$3^3\mid +a_0a_1a_2\mid 012\mid +a_0a_1a_3\mid 013\mid +a_0a_2a_3\mid 023\mid +a_1a_2a_3\mid 123\mid)^*.$$

(2). *Aronhold's Operator and the Identical Relations.*

Applying Aronhold's operators and collecting the coefficients of

$$a_0^2,\ a_1^2,\ a_2^2,\ a_3^2,\ a_0a_1,\ a_0a_2,\ a_0a_3,\ a_1a_2,\ a_1a_3,\ a_2a_3,$$

we have, putting each equal to zero:

$$3b_0\mid 0^3\mid +b_1\mid 0^21\mid +b_2\mid 0^22\mid +b_3\mid 0^23\mid \equiv 0,\ \text{coefficient of } a_0^2,$$

$$b_0\mid 01^2\mid +3b_1\mid 1^3\mid +b_2\mid 1^22\mid +b_3\mid 1^23\mid \equiv 0,\ \text{coefficient of } a_1^2,$$

$$b_0\mid 02^2\mid +b_1\mid 12^2\mid +3b_2\mid 2^3\mid +b_3\mid 2^23\mid \equiv 0,\ \text{coefficient of } a_2^2,$$

$$b_0\mid 03^2\mid +b_1\mid 13^2\mid +b_2\mid 23^2\mid +3b_3\mid 3^3\mid \equiv 0,\ \text{coefficient of } a_3^2,$$

$$2b_0\mid 0^21\mid +2b_1\mid 01^2\mid +b_2\mid 012\mid +b_3\mid 013\mid \equiv 0,\ \text{coefficient of } a_0a_1,$$

$$2b_0\mid 0^22\mid +b_1\mid 012\mid +2b_2\mid 02^2\mid +b_3\mid 023\mid \equiv 0,\ \text{coefficient of } a_0a_2,$$

$$2b_0\mid 0^23\mid +b_1\mid 013\mid +b_2\mid 023\mid +2b_3\mid 03^2\mid \equiv 0,\ \text{coefficient of } a_0a_3,$$

$$b_0\mid 012\mid +2b_1\mid 1^22\mid +2b_2\mid 12^2\mid +b_3\mid 123\mid \equiv 0,\ \text{coefficient of } a_1a_2,$$

$$b_0\mid 013\mid +2b_1\mid 1^23\mid +b_2\mid 123\mid +2b_3\mid 13^2\mid \equiv 0,\ \text{coefficient of } a_1a_3,$$

$$b_0\mid 023\mid +b_1\mid 123\mid 2b_2\mid 2^23\mid +2b_3\mid 23^2\mid \equiv 0,\ \text{coefficient of } a_2a_3.$$

(3). *Analysis of the Operator into Three Operators.*

We may notice the formation of these equations. They contain three operators; an operator 0, 1, 2, 3 applied to the indices of the a's whose coefficient we seek, gives the combinations within the strokes; b_0, b_1, b_2, b_3 give the literal coefficients and when the exponent of one of the indices in the strokes exceeds the exponent of the same index of the a's, whose coefficient we seek, this exponent becomes the numerical coefficient of the term in question. *E. g.*, for the coefficient of a_2^2, we apply 0, 1, 2, 3, to $\mid 2^2\mid$ and get

*The coefficient of $a_ia_ja_j$ has been farther abbreviated to $\mid i\ k\ j\mid$ for obvious reasons.

$|\,02^2\,|$, $|\,12^2\,|$, $|\,2^3\,|$, $|\,2^23\,|$ since here $|\,i\ k\,| = |\,k\ i\,|$.

b_0, b_1, b_2, b_3, are the corresponding literal coefficients;

1, 1, 3, 1 are the corresponding numerical coefficients, and

$b_0\,|\,02^2\,| + b_1\,|\,12^2\,| + 3b_2\,|\,23^2\,| + b_3\,|\,2^23\,|$ is the coefficient of a_2^2. Again, for the coefficient of a_1a_3, 0, 1, 2, 3, to $|\,13\,|$ give

$|\,013\,|$, $|\,1^23\,|$, $|\,123\,|$, $|\,13^2\,|$ also we have

b_0, b_1, b_2, b_3, and

1, 2, 1, 2; finally

$b_0\,|\,013\,| + 2b_1\,|\,1^23\,| + b_2\,|\,123\,| + 2b_3\,|\,13^2\,|$ = the coefficient of a_1a_3.

(4). *Determinant Relations between Symmetric Functions.*

From the first four equations of (2) we have by elimination of b_0, b_1, b_2, b_3

$$\begin{vmatrix} 3\,|\,0^3\,| & |\,0^21\,| & |\,0^22\,| & |\,0^23\,| \\ |\,01^2\,| & 3\,|\,1^3\,| & |\,1^22\,| & |\,1^23\,| \\ |\,02^2\,| & |\,12^2\,| & 3\,|\,2^3\,| & |\,2^23\,| \\ |\,03^2\,| & |\,13^2\,| & |\,23^3\,| & 3\,|\,3^3\,| \end{vmatrix} \equiv 0.$$

In a similar way, using four equations at a time of the ten, we should have $\frac{10.9.8.7}{1.2.3.4} = 210$ identically vanishing determinant relations between the symmetric functions involved.

(5). *The Calculation of the Functions by the Identical Equations.*

a. Of the stroked elements in the equations of (2) we know $|\,0^3\,|$, $|\,1^3\,|$, $|\,2^3\,|$, $|\,3^3\,|$, $|\,2^23\,|$, $|\,23^2\,|$. They are $b_0^3\Sigma(\beta_1\beta_2\beta_3)^3$, $b_0^3\Sigma(\beta_1\beta_2\beta_3)^2$, $b_0^3\Sigma(\beta_1\beta_2\beta_3)$, $b_0^3\Sigma(\beta_1\beta_2\beta_3)^0$, $b_0^3\Sigma\beta_1\beta_2$, $b_0^3\Sigma\beta_1$, equal to $-b_3^3$, $b_0b_3^2$, $-b_0^2b_3$, b_0^3, $b_0^2b_2$, $-b_0^2b_1$, respectively.

b. We may use them for solving the functions as follows: The equations of (2) become in order,

$$3b_0^4\Sigma(\beta_1\beta_2\beta_3)^3 + b_1b_0^3\Sigma(\beta_1\beta_2\beta_3)^2 b_0\Sigma\beta_1\beta_2 + b_2b_0\Sigma\beta_1\beta_2\beta_3 b_0^2\Sigma\beta_1^2\beta_2^2$$

$$+ b_0^3b_3\Sigma\beta_1^3\beta_2^3 = 0, \text{ or } -3b_0b_3^3 + b_1b_2b_3^2 - b_2^2b_0^2\Sigma\beta_1^2\beta_2^2 + b_3b_0^3\Sigma\beta_1^3\beta_2^3 = 0,$$

and in a similar way, $-b_0b_1b_3^2 + 3b_0b_1b_3^2 - b_0b_2^2b_3 + b_3b_0^3\Sigma\beta_1^2\beta_2^2 = 0$. These two give $b_0^3\Sigma\beta_1^2\beta_2^2$ and $b_0^2\Sigma\beta_1^2\beta_2^2$. The third equation gives

$$b_0b_0^3\Sigma\beta_1^3\beta_2\beta_3 + b_1b_0^3\Sigma\beta_1^2\beta_2\beta_3 + 3b_2b_0^3\beta_1\beta_2\beta_3 + b_3b_0^3\Sigma\beta_1\beta_2 = 0, \text{ or}$$

$$-b_0b_3b_0^2\Sigma\beta_1^2 + b_0b_1^2b_3 - 3b_0^2b_2b_3 + b_0^2b_2b_3 = 0.$$

It gives $\Sigma\beta_1^2$, but that is known if the previous resultant is calculated. The next equation gives $b_0^4\Sigma\beta_1^3 + b_1b_0^3\Sigma\beta_1^2 - b_0^2b_1b_2 + 3b_0^3b_3 = 0$.

With the preceding it gives $b_0^3\Sigma\beta_1^3$.

[To be continued.]

TEOREMA.

By J. M. MONSANTO, Mayaguez, Porto Rico, W. I.

Encontré en un libro, que el cubo de un número menos su raiz, es un multiplo de 6, es decir que $x^3-x=6n$, pero no lo demostraba. Hallé la demostracion y traté de sacar algun partido de esté teorema. Nada encontré, pero en mis investigaciones di con otro teorema que he aplicado a la extracion de la raiz cubica.

El teorema es el siguiente: Todo número dividida por 6, dá un residuo igual al que dá su cubo dividido por dicho número 6, o vice versa,—todo cubo dividido por 6, da un residuo igual al que da su raiz dividida por dicho número 6. Efectivamente, todo número que no es multiple de 6, puede representarse por una de las siguientes espresiones—$x+1$, $x+2$, $x+3$, $x+4$, $x+5$, siendo x igual a cero ó un multiplo de 6.

Si x es igual 0, hallaremos que los cubos de 1, 2, 3, 4, 5 ó 1, 8, 27, 64, 125, divididos por 6, dan por residuo 1, 2, 3, 4, 5.

Si x es un multiplo de 6, tendremos $(x+1)^3=x^3+3x^2+3x+1$, expresion que dividida por 6 dá por residuo 1.

$(x+2)^3=x^3+6x^2+12x+8$, expresion que dividida por 6, nos dá un residuo de 2. Asi mismo se encuentra que $\left.\begin{matrix}(x+3)^3\\(x+4)^3\\(x+5)^3\end{matrix}\right\}$ divididos por 6 dan por residuo, 3, 4, 5, y queda demonstrado el teorema.

La extraccion de la raiz cubica es una operacion algo dificil y que exige bastantes cálculos; asi es que algunos tratados elementales de aritmetica aconsejan que para la extraccion de dichas raices se proceda por tanteo, pero el teorema arriba indicado permite reducir este tanteo á limites muy estrechos tratandose de cubos perfectos. Conociendo el número final de este cubo, desde luego sabremos en que número termina su raiz, y restando de este número el residuo que da la division del cubo por 6, podremos saber en que número hader terminar el multiplo de 6, que agregado al residuo, da la raiz.

Supongamos que se pida la raiz cubica del número 493039. Este cubo acaba en 9; por consiquiente su raiz debe acabar en 9. El residuo de la division por 6 es uno, 1, y $9-1=8$, es decir que la raiz debe ser un multiplo de 6 que acaba en 8 mas 1. 493039 es visiblemente menor que el cubo de 80 y mayor que el cubo de 70: entre 70 y 80, no hay mas que un multiplo de 6 que acaba en 8, que es 78 y desde luego digo sin buscar mas que la raiz es $78+1=79$.

Busquemos la raiz de 35,287,552. La raiz debe acabar en 8. Siendo el residuo de la division por 6, 4, la raiz debera ser formada por un multiplo de 6 que acabe en $4+4$. El dicho cubo es mayor que 300 y menor que 350 como se puede ver por una pequeña multiplicacion. Entre estos dos números no hay mas que un multiplo de 6 que acaba en 4. Es el número 324 y sin mas tanteos digo que la raiz es $324+4=328$.

THEOREM.

I found in a book, that the cube of a number less its root, is a multiple of 6, that is to say, that $x^3-x=6n$, but it was not demonstrated. I found the demonstration and tried to get some practical use out of this theorem. I found nothing, but in my investigations I found another theorem which I have applied to the extraction of the cube root. The theorem is the following: Every number divided by 6 gives a remainder equal to that which its cube gives, divided by said number 6; or vice versa, every cube divided by 6 gives a remainder equal to that which its cube root gives, divided by said number 6. Therefore, every number which is not a multiple of 6 may be represented by one of the following expressions: $x+1$, $x+2$, $x+3$, $x+4$, $x+5$, x being equal to zero or a multiple of 6. If x is equal to 0 we will find that the cubes of 1, 2, 3, 4, 5, or 1, 8, 27, 64, 125 divided by 6 give for remainders 1, 2, 3, 4, 5.

If x is a multiple of 6, we will have $(x+1)^3=x^3+3x^2+3x+1$, an expression that divided by 6, gives 1 for a remainder.

$(x+2)^3=x^3+6x^2+12x+8$, an expression that divided by 6, gives us 2 for a remainder. In the same way it is found that $\left.\begin{matrix}(x+3)^3\\(x+4)^3\\(x+5)^3\end{matrix}\right\}$ divided by 6 gives for remainders 3, 4, 5 (respectively), and the theorem is demonstrated.

The extraction of the cube root is a somewhat difficult operation, and (one) which requires much calculation; thus it is that some elementary treatises of arithmetic advise that for the extraction of said cubic roots (one) should proceed by approximation, but the theorem above indicated, allows (one) to reduce this approximation to very narrow limits when treating of perfect cubes. Knowing the final number of this cube, consequently we will know in what number its root ends, and subtracting from this number the remainder which the cube divided by 6 gives, we will be able to know in what number the multiple of 6 must end which, added to the remainder, gives the root.

Let us suppose that the cube root of 493039 is asked. This cube ends in 9, consequently its root must end in 9. The remainder of the division by 6 is 1, and 9--1=8, that is to say that the root must be a multiple of 6 which ends in 8, plus 1. 493039 is plainly less than the cube of 80 and more than the cube of 70; between 70 and 80 there is but one multiple of 6 which ends in 8, which is 78, and consequently I say without any more search that the root is 78+1=79.

Let us try to find the root of 35,287,552. The root must end by 8. Four, 4, being the remainder of the division by 6, the root must be formed of a multiple of 6 which ends in 4 + 4. The said cube is greater than 300, and less than 350, as may be seen by a small multiplication. Between these two numbers there is but one multiple of 6 which ends in 4. It is the number 324, and without more search I say that the root is 324+4=328.

[NOTE. This theorem and its translation was furnished by Dr. Halsted. EDITOR.]

NEW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

By BENJAMIN F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Curry University, Pittsburg, Pennsylvania.

[Continued from March Number.]

The following "dissection proofs" differ from the preceding ones only in the fact that there is a displacement of one or more of the squares from their usual places.

LXXXIII. Fig. 34.
Let the $\triangle ABC$ be right-angled at C.
$\triangle LHK = \triangle ABC$, and $\triangle LAE = \triangle HBF$.
$\therefore ABHL \backsimeq ACDE + DFHK$. Q. E. D.

LXXXIV. Fig. 34.
$LMOA \backsimeq LKCA \backsimeq ACDE$.
$HMOB \backsimeq HKCB \backsimeq HKDF$.
$\therefore ABHL \backsimeq ACDE + DFHK$. Q. E. D.

LXXXV. Fig. 34.
$ANQB \backsimeq AEFB \backsimeq AEDC$.
$LNQH \backsimeq LEFH \backsimeq FHKD$.
$\therefore ABHL \backsimeq ACDE + DFHK$. Q. E. D.

Fig. 34.

NOTE. In Fig. 34, if ALE is taken as the triangle, we have another type of figure, thus giving several more proofs. See Halsted's *Elements of Geometry*, page 78.

LXXXVI. Fig. 35.
The given triangle is ABC.
$\triangle LNO = \triangle APE'$. $\triangle LMO = \triangle ABC$.
$\triangle AHM = \triangle MDA \backsimeq PADE + BNK$.
$\therefore AMLB \backsimeq ADEE' + AHKC$. Q. E. D.

LXXXVII. Fig. 35.
$LRSB \backsimeq LOCB \backsimeq ADEE'$.
$AMRS \backsimeq AMOC \backsimeq AHKC$.
$\therefore AMLB \backsimeq ADEE' + AHKC$. Q. E. D.

LXXXVIII. Fig. 35.
$ABTX \backsimeq ABVD \backsimeq ADEE'$.
$MLTX \backsimeq MLVD \backsimeq AHKC$.
$\therefore AMLB \backsimeq ADEE' + AHKC$. Q. E. D.

Fig. 35.

NOTE. In Fig. 35, if AMH, MLO, or LBV is taken as the given triangle, we have slightly different types of figures, each yielding various proofs.

LXXXIX. Fig. 36.
$AOC = APE$. $MNB = HFB$.
$MNOL = HKPB$.
$\therefore ABML = ACDE + DFHK$. Q. E. D.

Fig. 36.

XC. Fig. 37.
$PBR = SBK$. $RPLO = ETAD$.
$LNO = ATF$. $NMAC = SBAH$.
$\therefore ABLM = ADEF + AHKC$. Q. E. D.

It will be observed that the above proofs differ chiefly in the relative position of the triangle and the squares. In order to give an idea of still other possible varieties, it may be noted for example, that instead of the arrangement as we have it it Fig. 35, the given triangle may have seven other positions within the square $ACKH$, right angles coinciding. Furthermore, in each case, the square on the hypotenuse may be constructed outwardly from the triangle, or overlapping it. One of these cases, it will be remembered, was considered in November number, 1897.

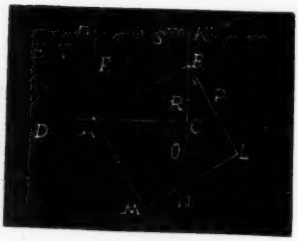


Fig. 37.

[To be continued.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

104. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Iowa.

If I should buy goods at a price 20% higher than I did buy them, and sell the goods for the same amount that I did sell them, I would gain 25% less than I did gain. What per cent. did I gain? (Solve by Arithmetic).

I. Solution by W. F. BRADBURY, Head Master, Cambridge Latin School, Cambridge, Mass.

If I gain 25% less, I get $\frac{4}{5} \times \frac{25}{100}$, or $\frac{1}{500}$ of \$1 less than if I sold at the same per cent. advance. If I sell at same per cent. advance, I should receive $\frac{4}{5}(1+\frac{r}{100}) = \frac{4}{5} + \frac{6r}{500}$. What I did receive was $1+\frac{r}{100}$. Subtract, and we have

$\frac{1}{5}+\frac{r}{500}$, loss, or $\frac{1}{5}+\frac{r}{500}=\frac{4}{5}\frac{2}{0}\frac{0}{0}$, $\frac{r}{500}=\frac{5}{5}\frac{0}{0}\frac{0}{0}$, or $r=50$.

Do you call this an Arithmetical solution? Instead of r, suppose you write "*same*"? Pure Algebra: $1+\frac{r}{100}=\frac{6}{5}\left(1+\frac{r}{100}-\frac{2}{1}\frac{5}{0}\frac{}{0}\right)$. Then $r=50$.

[NOTE.—According to our definition of an algebraical solution, No. 5, Vol. V., page 139, of the MONTHLY, Professor Bradbury's solution is algebraic. It is immaterial as to what sort of a character is used to represent the quantity sought. It may be a letter, a character of any kind, a word, or several words.

The definition referred to, viz.: Any solution in which the result sought is represented by some character, which character is operated upon until the condition or conditions of the problem are fulfilled which condition or conditions are stated in the form of an equation from which the numerical value of the character is determined, is an algebraic solution,—has received the sanction of some of the best mathematicians in this country. EDITOR F.]

II. Solution by J. OWEN MAHONEY, B. E., Mc., Professor of Mathematics and Science, Carthage Graded and High School, Carthage, Tex.; J. W. YOUNG, Columbus, O.; ELMER SCHUYLER, High Bridge, N. J.; WALTER H. DRANE, Graduate Student, Harvard University, 65 Hammond Street, Cambridge, Mass.; and the PROPOSER.

Since 100% of the purchasing price is paid for the goods in the first instance, it is easily seen that

the % gain$-20\%:120::$ the % gain$-25\%:100$,

or the % gain$-20\%:5\%::120:20$

or the % gain$-20\%:5\%::6:1$.

$\therefore$ The % gain$=50$.

III. Solution by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

1. Let 100%=actual cost, then
2. 120%=supposed cost.
3. Let *100%*=selling price. [The italics are used for distinction].
4. *100%*-100%=actual gain, and
5. *100%*-120%=supposed gain.
6. $\frac{100\%-100\%}{100\%}-\frac{100\%-120\%}{120\%}=\frac{1}{6}$, or $\frac{100\%-100\%}{100\%}-\frac{83\frac{1}{3}\%-100\%}{100\%}=\frac{1}{6}$,

or $\frac{16\frac{2}{3}\%}{100\%}=\frac{1}{6}$, whence

7. $16\frac{2}{3}\%=25\%$; $1\%=1.5\%$; $100\%=150\%$, selling price in terms of cost price.
8. $\therefore$ $150\%-100\%=50\%$, the gain %.

Also solved by *G. B. M. ZERR* and *B. F. YANNEY*.

105. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Iowa.

A teacher looks at his watch when leaving school at noon. When he comes back he finds that the hour hand and the minute hand have just changed places (that they had when he left the school). What time was it when he left, and what time when he came back to school? (Solve by Arithmetic.)

I. Solution by JOHN M. ARNOLD, Crompton, R. I.

During the time that the teacher was away from the school, the distance traveled by the minute hand added to the distance traveled by the hour hand,

LXXXIX. Fig. 36.
$AOC = APE$. $MNB = HFB$.
$MNOL = HKPB$.
$\therefore ABML \backsimeq ACDE + DFHK$. Q. E. D.

XC. Fig. 37.
$PBR = SBK$. $RPLO = ETAD$.
$LNO = ATF$. $NMAC = SBAH$.
$\therefore ABLM \backsimeq ADEF + AHKC$. Q. E. D.

Fig. 36.

It will be observed that the above proofs differ chiefly in the relative position of the triangle and the squares. In order to give an idea of still other possible varieties, it may be noted for example, that instead of the arrangement as we have it it Fig. 35, the given triangle may have seven other positions within the square $ACKH$, right angles coinciding. Furthermore, in each case, the square on the hypotenuse may be constructed outwardly from the triangle, or overlapping it. One of these cases, it will be remembered, was considered in November number, 1897.

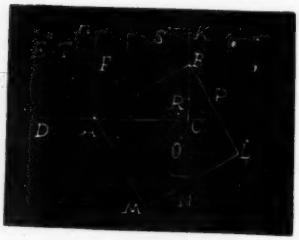


Fig. 37.

[To be continued.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

104. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Iowa.

If I should buy goods at a price 20% higher than I did buy them, and sell the goods for the same amount that I did sell them, I would gain 25% less than I did gain. What per cent. did I gain? (Solve by Arithmetic).

I. Solution by W. F. DRADBURY, Head Master, Cambridge Latin School, Cambridge, Mass.

If I gain 25% less, I get $\frac{6}{8} \times \frac{25}{100}$, or $\frac{150}{800}$ of $1 less than if I sold at the same per cent. advance. If I sell at same per cent. advance, I should receive $\frac{6}{8}(1+\frac{r}{100}) = \frac{6}{8} + \frac{6r}{800}$. What I did receive was $1+\frac{r}{100}$. Subtract, and we have

$\frac{1}{5}+\frac{r}{500}$ loss, or $\frac{1}{5}+\frac{r}{500}=\frac{4}{5}\frac{5}{0}\frac{0}{0}$, $\frac{r}{500}=\frac{5}{5}\frac{0}{0}\frac{0}{0}$, or $r=50$.

Do you call this an Arithmetical solution? Instead of r, suppose you write "*same*"? Pure Algebra: $1+\frac{r}{100}=\frac{6}{5}\left(1+\frac{r}{100}-\frac{2}{1}\frac{5}{0}\frac{0}{0}\right)$. Then $r=50$.

[NOTE.—According to our definition of an algebraical solution, No. 5, Vol. V., page 139, of the MONTHLY, Professor Bradbury's solution is algebraic. It is immaterial as to what sort of a character is used to represent the quantity sought. It may be a letter, a character of any kind, a word, or several words.

The definition referred to, viz.: Any solution in which the result sought is represented by some character, which character is operated upon until the condition or conditions of the problem are fulfilled which condition or conditions are stated in the form of an equation from which the numerical value of the character is determined, is an algebraic solution,—has received the sanction of some of the best mathematicians in this country. EDITOR F.]

II. Solution by J. OWEN MAHONEY, B. E., Mc., Professor of Mathematics and Science, Carthage Graded and High School, Carthage, Tex.; J. W. YOUNG, Columbus, O.; ELMER SCHUYLER, High Bridge, N. J.; WALTER H. DRANE, Graduate Student, Harvard University, 65 Hammond Street, Cambridge, Mass.; and the PROPOSER.

Since 100% of the purchasing price is paid for the goods in the first instance, it is easily seen that

the % gain—20% : 120 :: the % gain—25% : 100,

or the % gain—20% : 5% :: 120 : 20

or the % gain—20% : 5% :: 6 : 1.

∴ The % gain = 50.

III. Solution by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

1. Let 100% = actual cost, then
2. 120% = supposed cost.
3. Let *100*% = selling price. [The italics are used for distinction].
4. *100*% − 100% = actual gain, and
5. *100*% − 120% = supposed gain.
6. $\frac{100\%-100\%}{100\%}-\frac{100\%-120\%}{120\%}=\frac{1}{4}$, or $\frac{100\%-100\%}{100\%}-\frac{83\frac{1}{3}\%-100\%}{100\%}=\frac{1}{4}$, or $\frac{16\frac{2}{3}\%}{100\%}=\frac{1}{4}$, whence
7. $16\frac{2}{3}\%=25\%$; $1\%=1.5\%$; $100\%=150\%$, selling price in terms of cost price.
8. ∴ 150% − 100% = 50%, the gain %.

Also solved by *G. B. M. ZERR* and *B. F. YANNEY*.

105. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Iowa.

A teacher looks at his watch when leaving school at noon. When he comes back he finds that the hour hand and the minute hand have just changed places (that they had when he left the school). What time was it when he left, and what time when he came back to school? (Solve by Arithmetic.)

I. Solution by JOHN M. ARNOLD, Crompton, R. I.

During the time that the teacher was away from the school, the distance traveled by the minute hand added to the distance traveled by the hour hand,

would make the whole circle of the dial. As the minute hand moves twelve times as fast as the hour hand the spaces passed over would be $\frac{12}{13}$ and $\frac{1}{13}$ of a revolution, respectively.

$\frac{12}{13}$ of a revolution of the minute hand equals $55\frac{5}{13}$ minutes, the time the teacher was away. When the teacher left, the distance between the hands was $\frac{11}{13}$ of the distance of the minute hand from the zero point. Hence $\frac{1}{13}\times\frac{12}{11}=\frac{12}{143}$ $=5\frac{5}{143}$ minutes past twelve, the time when he left. Add the time that he was away, $5\frac{5}{143}+55\frac{5}{13}=60\frac{60}{143}$ or $\frac{60}{143}$ of a minute past one, the time when he returned.

II. Solution by BENJAMIN F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, O.

We first solve the general problem: At what times are the positions of the hands of a watch interchanged?

It is plain that there are some positions not interchangeable, as, for instance, minute-hand at 6 and hour-hand midway between any two consecutive numbers.

It is evident, also, that in the case of any possible position of interchange, in going from one position to the interchanged one, the two hands must together travel over $60n$ minute spaces, n being an integer.

Then, since the minute-hand travels 12 times as fast as the hour-hand, $\frac{1}{13}$ of the distance traversed by both, or $60n/13$ minute spaces, must be the distance traveled by the hour hand in going from one position of interchange to the other, and $\frac{12.60n}{13}$ minute-spaces, the minute-hand's distance. Furthermore, $60n/13$ minute-spaces is the distance in any case of interchange, from hour-hand to minute hand, always reckoned clockwise.

We now have the relative positions of the hands with respect to each other. We next proceed to find the times of these relative and interchanged positions. Suppose the hands at 12. In order that they shall be $60n/13$ minute-spaces apart, the minute-hand must travel over $\frac{12.60n}{11.13}$ minute-spaces. This, the number of minute-spaces past 12, gives the time of the first of any two interchanged positions, and $\frac{12.60n}{11.13}+\frac{12.60n}{13}$ is the time of the corresponding second position.

Now starting, say, with 12 o'clock noon, and substituting for n in succession 1, 2, 3, ...143, which completes a cycle, we shall find, omitting the cases in which the hands are together, 132 different answers to the general problem. We give a few:

$5\frac{5}{143}$ minutes past 12 noon, and $\frac{60}{143}$ of a minute past 1 P. M.
$10\frac{10}{143}$ " " " " " $\frac{120}{143}$ " " " 2 P. M.
....
$\frac{60}{143}$ of a minute past 1 P. M., and $5\frac{5}{143}$ minutes past 12 A. M.
.....
$59\frac{83}{143}$ minutes past 10 P. M., and $54\frac{138}{143}$ minutes past 11 P. M.

The first two given are possible answers to the special case in hand.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; B. F. SINE, Principal Capon Bridge Normal School; Capon Bridge, W. Va.; J. D. CRAIG, Frankfort, N. J.; and MARTIN SPINX, Wilmington, O.

Since the two hands had precisely changed positions, they together had passed over all the spaces on the dial-face ; but, as the minute-hand always goes through 60 spaces while the hour-hand goes through 5, both go through 65.

$\therefore$ $65:5=60:4\frac{8}{13}$.

$\therefore$ $4\frac{8}{13}$ spaces is the number of spaces passed over by the hour-hand. This is also the distance the minute-hand was in advance of the hour-hand in the first position.

Since the time he left at noon was after 12 o'clock and since the minute-hand always gains 55 minutes in 60 minutes, to gain $4\frac{8}{13}$ minutes we have $55:60=4\frac{8}{13}:5\frac{5}{143}$.

$\therefore$ The time was $5\frac{5}{143}$ minutes after 12 o'clock. In the second position, the hour-hand was $4\frac{8}{13}$ minutes in advance of the minute-hand. $5\frac{5}{143}-4\frac{8}{13}=\frac{60}{143}$ minutes.

$\therefore$ The time was $\frac{60}{143}$ minutes after 1 o'clock.

$\therefore$ He left at 5 minutes $2\frac{14}{143}$ seconds after 12 o'clock, and returned at $25\frac{36}{143}$ seconds after 1 o'clock.

Also solved by *W. F. BRADBURY, J.W. YOUNG, WALTER H. DRANE, ELMER SCHUYLER*, and *ALOIS F. KOVARIK*.

ALGEBRA.

89. Proposed by G. A. MILLER, Ph. D., Instructor in Mathematics, Cornell University, Ithica, N. Y.

Solve by quadratics, $x^2+y=7$.....(1),
$x+y^2=11$.....(2).

XI. Solution by W. A. HARSHBARGER, A. M., Professor of Mathematics, Washburn College, Topeka, Kas.

$y^2+x=11$.....(1), $y+x^2=7$.....(2).

(1)$-$(2) $(y^2-x^2)-(y-x)=4$.....(3).

Put $(y+x)=a$, and $(y-x)=b$.

Then by substituting in (3), $ab-b=4$.....(4).

$\therefore$ $a^2b^2=16+8b+b^2$.....(5).

Subtract, $10ab=40+10b$.

$\therefore$ $a^2b^2-10ab=-24-2b+b^2$.....(6), and

$a^2b^2-10ab+25=1-2b+b^2$.....(7).

$\therefore$ $ab-5=1-b$, $ab+b=6$.....(8). (4)$+$(8), $ab=5$; (4)$-$(8), $b=1$.

$\therefore$ $a=5$. $\therefore$ $y+x=5$, and $y-x=1$. $\therefore$ $x=2$, and $y=3$.

[NOTE. Professor Harshbarger says the above solution appeared in one of the scientific journals a few years ago, but he has forgotten the name of the author.]

90. Proposed by J. MASCUS BOORMAN, Consultative Mechanician and Counselor at Law, Woodmere, Long Island, N. Y.

Solve $x^2+y^2+m(x+y)=m^2$.....(1).

$x^2+y^2+xy=m^2$.....(2).

I. Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; A. H. BELL, Hillsboro, Ill.; HON. JOSIAH H. DRUMMOND, LL. D., Portland, Me.; CHARLES C. CROSS, Libertytown, Md.; ELMER SCHUYLER, High Bridge, N. J.; and M. A. GRUBER, A. M., Washington, D. C.

Subtracting we have $m(x+y)=xy$.....(3).

Squaring (3) $m^2(x^2+2xy+y^2)=x^2y^2$.

Multiplying (2) by m^2, $m^2(x^2+xy+y^2)=m^4$.

Whence $x^2y^2-m^2xy=m^4$.....(4),

Solving as a quadratic, $xy=\frac{m^2}{2}(1\pm\sqrt{5})$.....(5).

Adding (5) and (2), $x^2+2xy+y^2=\frac{m^2}{4}(6\pm2\sqrt{5})$.

Whence $x+y=\pm\frac{m}{2}(\sqrt{5}\pm1)$.

Similarly, $x-y=\pm\frac{m}{2}(\sqrt{-2\mp6\sqrt{5}})$.

Whence $x=\pm\frac{m}{4}(\sqrt{5}\pm1+\sqrt{-2\mp6\sqrt{5}})$,

$y=\pm\frac{m}{4}(\sqrt{5}\pm1-\sqrt{-2\mp6\sqrt{5}})$,

II. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

(1)−(2) gives $m(x+y)=xy$. Whence $x+y=xy/m$.

Put $x=vy$. Then $vy+y=vy^2/m$, and $v+1=vy/m$.

Whence $y=m(v+1)/v$, and $x=vy=m(v+1)$.

Substituting the values of x and y in (1) or (2), we have

$$m^2(v+1)^2+\frac{m^2(v+1)^2}{v^2}+\frac{m^2(v+1)^2}{v}=m^2.$$

Dividing by m^2, clearing of fractions, expanding, etc., we obtain

$$v^4+3v^3+4v^2+3v+1=v^2.$$

Adding and subtracting v^3+2v^2+v, we find

$$(v+1)^4-v(v+1)^2=v^2.$$

Completing square and extracting square root, we get

$$2(v+1)^2-v=\pm v\sqrt{5}.$$

Expanding, transposing, and uniting terms, we obtain

$$2v^2+(3\mp\sqrt{5})v=-2.$$

Completing square, extracting square root, etc., we find

$$v=\tfrac{1}{4}[-3\pm\sqrt{5}\pm\sqrt{-2(1\pm3\sqrt{5})}].$$

Whence $v+1=\tfrac{1}{4}[1\pm\sqrt{5}\pm\sqrt{-2(1\pm3\sqrt{5})}]$,

and $(v+1)/v=\tfrac{1}{4}[1\pm\sqrt{5}\mp\sqrt{-2(1\pm3\sqrt{5})}]$.

$\therefore x=m(v+1)$ and $y=m[(v+1)/v]$.

III. Solution by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa., and ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Iowa.

Let $x=v+z$, and $y=v-z$. $2v^2+2z^2+2mv=m^2$. $3v^2+z^2=m^2$.

Whence $v=\tfrac{1}{4}(1\pm\sqrt{5})m$, and $z=\pm\tfrac{1}{4}\sqrt{(-2\mp6\sqrt{5})}m$.

The four values of x and y are

$x=\tfrac{1}{4}[1+\sqrt{5}+\sqrt{(-2-6\sqrt{5})}]m$; $y=\tfrac{1}{4}[1+\sqrt{5}-\sqrt{(-2-6\sqrt{5})}]m$.

$x=\tfrac{1}{4}[1+\sqrt{5}-\sqrt{(-2-6\sqrt{5})}]m$; $y=\tfrac{1}{4}[1+\sqrt{5}+\sqrt{(-2-6\sqrt{5})}]m$.

$x=\tfrac{1}{4}[1-\sqrt{5}+\sqrt{(-2+6\sqrt{5})}]m$; $y=\tfrac{1}{4}[1-\sqrt{5}-\sqrt{(-2+6\sqrt{5})}]m$.

$x=\tfrac{1}{4}[1-\sqrt{5}-\sqrt{(-2+6\sqrt{5})}]m$; $y=\tfrac{1}{4}[1-\sqrt{5}+\sqrt{(-2+6\sqrt{5})}]m$.

IV. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa., and J. SCHEFFER, A. M., Hagerstown, Md.

Let $x+y=t$, $xy=r$.

Then (1) and (2) become $t^2-2r+mt=m^2$.....(3), $t^2-r=m^2$.....(4).

Subtracting (4) from (3) we get $r=mt$.....(5), or $t=r/m$.....(6).

(5) in (3) gives $t^2-mt=m^2$. $\therefore t=\tfrac{1}{2}m(1\pm\sqrt{5})$.

(6) in (4) gives $r^2-m^2r=m^4$. $\therefore r=\tfrac{1}{2}m^2(1\pm\sqrt{5})$.

Since $x+y=t$ and $xy=r$, we have

$x=\tfrac{1}{2}[t\pm\sqrt{(t^2-4r)}]=\tfrac{1}{4}m\{1\pm\sqrt{5}\pm\sqrt{[-2(1\pm3\sqrt{5})]}\}$.

$y=\tfrac{1}{2}[t\mp\sqrt{(t^2-4r)}]=\tfrac{1}{4}m\{1\pm\sqrt{5}\mp\sqrt{[-2(1\pm3\sqrt{5})]}\}$.

V. Solution by the PROPOSER.

x/m, y/m ; with *eight* roots (four each) [or sixteen *by* $\sqrt{m^2}=\pm m$] ; $x:y=$ ratio $X:Y$ with values $x=X$ times *any-* or *every-*thing ; $y=Y$ *any-* or *every-*thing ; $\therefore x/X=y/Y=m$ of an infinite degree and roots, *all* values from $\pm(1/\infty^{\infty})$ to 0 to $\pm\infty^{\infty}$. And that m is a *blind* factor in $x=Xm$; $y=Ym$;—SOLVE CIRODE'S curious PROBLEM. (I)—(II).....$m(x+y)=xy$.....(III). *Un*-factor $x=Xm$; $y=Ym$.....(A). $\therefore X^2m^2+Y^2m^2+m(Xm+Ym)=m^2$.....($I_1$) ; $m(Xm+Ym)=XmYm$.....(III_1) *divided* by m^2 are $X^2+Y^2+x+y=1$..... (I_a) ;

$X+Y=XY$.....(III$_a$). *Twice* (III$_a$)+(I$_a$) gives $(X+Y)^2-(X+Y)=1$..... (IV). $\therefore X+Y=\frac{1}{2}(1\pm\sqrt{5})$.....(V). $\therefore$ By (III$_a$) $XY=\frac{1}{2}(1\pm5)$.....(VI) $\therefore$ (V)(VI), (by *nature*) the constants of $X^2-\frac{1}{2}(1\pm\sqrt{5})X+\frac{1}{2}(1\pm\sqrt{5})=0$..... (VII) whose *four* roots are by *its origin* (by *turns*) the *eight* roots of *both* X and Y. Multiplying (VII+)(VII−); is $X^4-X^3+2X=0$.....(B) *two* real roots each..... $\left\{\begin{matrix} X=\frac{1}{4}[1-\sqrt{5}+\sqrt{(-2+6\sqrt{5})}]=Y_1 \\ X_1=\frac{1}{4}[1-\sqrt{5}+\sqrt{(-2+6\sqrt{5})}]=Y \end{matrix}\right\}$.....(VIII). *Two* imaginery each..... $\left\{\begin{matrix} X_2=\frac{1}{4}[1-\sqrt{5}+\sqrt{(-1-6\sqrt{5})}]=Y_3 \\ X_3=\frac{1}{4}[1+\sqrt{5}-\sqrt{(-2-6\sqrt{5})}]=Y_2 \end{matrix}\right\}$ = 0,809016,99437 ±3.926373,373101,2946$\sqrt{-1}$.....(IX).

$\therefore \left\{\begin{matrix} X=-1.153721,375541,766 \text{ to } Y=0.535687..... ; \\ X_1=0.535687,386791,872 \text{ to } Y_1=-1.153721..... ; \end{matrix}\right.$

$$\left.\begin{matrix} X_2=.809016,99.....+3.926373,..... \sqrt{-1}=Y_3 \\ X_3=.809016,99.....-3.926373,.....\sqrt{-1}=Y_2 \end{matrix}\right\} ..... (C).$$

$\therefore X+Y=X_1+Y_1=-0.618033,988750-$; $X_2+Y_2=1.618033,988750-=X_3+Y_3$.....(D). *Proof* by $XY=X+Y$.....(III$_a$) $\left\{\begin{matrix} X=-1.15372137..... \\ Y=0.53568738..... \end{matrix}\right\}$

$$\left.\begin{matrix} \text{Product}=-0.6180339,88+=XY \\ \text{Sum} \quad =-0.6180339,89-=X+Y \end{matrix}\right\} .....(XI). \text{ Also (C) and (A).}$$

$\left\{\begin{matrix} x=Ym=m(-1.1537..... \\ y=Ym=m(\ 0.5356..... \end{matrix}\right\}$ Product, $mXmY=xy=in\ x$ Sum, $mX+mY=x+y$

$$\left.\begin{matrix} y=m_+(x+y)=m^2(-0.61803.....) \\ \therefore \text{ Multiplying}=m=m(x+y)=m^2(-0.61802\ .....) \end{matrix}\right\} ..... (XII).$$

*because (III)..... $xy=m(x+y)=m^2(-0.618033.98.....)=m^2XY$.....(E). $\therefore m$ (*vanishes*) is only a *blind* factor in x and y. $\therefore$ by (E) ratio $x:y=X:Y$ and and (I) (II) are *fully* solved, process, ratio and roots. Q. E. D. [As, solve (III) for m. $\therefore m=\frac{1}{2}(-1\mp\sqrt{5})(x+y)$..... *i. e.* $1=\frac{1}{2}(-1\mp\sqrt{5})(X+Y)$..... $\therefore m$ and x and y vanish together as *always*].

Corollary. Two quadratics in *three* unknowns, *if* their every term be quadrate (in unknowns) stand for, *and solve*, the two bi-quadratics each in one *true* unknown found by treating two stated unknowns as if multiples (or other functions) of the third, and un-factoring them from it (into two *new* letters). This treatment also determines the third unknown, roots and equation, or their nature and forms when inexpressible.

Corollary 2. *Strictly* $\sqrt{m}=\pm m$. $\therefore x^2+y^2\pm(x+y)$.....?(I$_c$) with (II$_c$) have *eight* roots each, viz.: (VIII, IX) or (C) direct, and *also* with signs reversed.

Scholium. *General* form $x^2+y^2+c(x+y)=a$.....(A); $x^2+y^2+xy=b$ .....(*B*) solves (by quadratics) many biquadrates, decomposes surd compounds (other than of 1 with $\sqrt{5}$), finds roots, etc.

GEOMETRY.

106. Proposed by C. HORNUNG, A. M., Professor of Mathematics, Heidelberg University, Tiffin, Ohio.

Upon the sides of any triangle ABC let the equilateral triangles ABD, BCE, and CAF be described, and let their exterior sides produced intersect, BE and AF in K, DB and FC in L, and DA and EC in M. Prove DK, EL, FM, parallel.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; J. SCHEFFER, A. M., Hagerstown, Md., and G. I. HOPKINS, Instructor in Mathematics and Physics, High School, Manchester, N. H.

Angle $KAB=180°-60°-$angle CAB, angle $ABK=180°-60°-$angle ABC.

$\therefore$ Adding, angle $KAB+$angle $ABK=240°-$angle $CAB-$angle ABC. Angle $AKB=180°-$(angle $KAB+$ angle ABK).

$\therefore$ Angle $AKB=$angle $CAB+$angle $ABC-60°$.

Again, angle $BCL=180°-60°-$angle ACB. Angle $ACB=180°-$(angle $CAB+$angle ABC).

$\therefore$ Angle $BCL=$angle $CAB+$angle $ABC-60°$.

$\therefore$ Angle $BCL=$angle AKB.

Angle $MAC=60°+$angle MAF, and angle $KAB=60°+$angle KAD. $\therefore$ Angle $MAC=$angle KAB.

Similarly, angle $MCA=$angle BCL. $\therefore$ Angle $MCA=$angle AKB.

$\therefore$ Triangle AMC is similar to triangle AKB.

$\therefore$ $AC : AK :: AM : AB$, or $AF : AK :: AM : AD$.

$\therefore$ Triangle AKF is similar to triangle AKD.

$\therefore$ Angle $AMF=$angle ADK.

$\therefore$ KD is parallel to MF. Similarly EL is parallel to MF.

II. Solution by the PROPOSER.

Points K, F, C, B are concyclic. $\because \angle CBE=\angle AFC$.

$\therefore \angle BKA=\angle BCL$. $\angle LBC=\angle KBA$.

$\therefore$ Triangles AKB and CBL are similar, and $KB/AB=LB/CB$, or $KB/EB=LB/EB$.

$\therefore$ KD is parallel to EL.

Similarly $AM/AD=AD/AK$, and therefore FD is parallel to KD.

And therefore EL is parallel to KD is parallel to FM.

III. Solution by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pa.

Let ABC be any triangle having equilateral triangles described upon its sides, and their exterior sides produced to intersect BE and AF in K, DB and FC in L, and DA and EC in M. Join FM, DK and EL.

The triangles BCL and ACM are similar, hence $BC : CL :: CM : CA$, or $CE : CL :: CM : CF$. And since the $\angle ECL=\angle FCM$, the triangles CLE and CFM are similar and equiangular, the angle FMC being equal to the angle LEC.

$\therefore$ ER is parallel to FM.....(1).

The triangles ABK and AMC are similar, hence $AB : AK :: AM : AC$, or $AD : AK :: AM : AF$.

Since the $\angle DAK = \angle MAF$, the triangles DKA and MFA are similar, and $\angle ADK$ is equal to $\angle AMF$.

$\therefore DH$ is parallel to FM..... .....(2).

$\therefore DK$, FM and EL are parallel. Q. E. D.

IV. Solution by CHARLES C. CROSS, Libertytown, Md.

Draw the figure as indicated in the problem.

Let $\angle BLE = x$, $\angle CEL = y$, $\angle DKB = z$, $\angle ADK = w$, $\angle CEM = v$, and $\angle FMC = w$.

$\angle ECL = 180° - (120° + C) = 60° - C$.

Similarly, $\angle EBL = A - 60°$, and $\angle KAD = 60° - A$.

$\angle BCL = 180° - (60° + C) = 120° - C$.

Similarly, $\angle LBC = 120° - B$, and $\angle BAK = 120° - A$.

Hence $\angle BLC = B + C - 60°$, and $\angle BKA = B + A - 60°$.

$\angle BLE + \angle BLC + \angle CEL + \angle ECL = 180°$; by substitution $B + x + y = 180°$...(1).

$\angle BKA + \angle BKD + \angle KDA + \angle KAD = 180°$; by substitution $B + w + z = 180°$..(2).

From (1) and (2), $x + y = w + z$..... .....(3).

If EL and DK are parallel, angle DKB=angle BEL, and angle BLE= angle KDB, or $z = 60° + y$ and $x = 60° + w$. Substituting in (3), $60° + w + y = 60° + w + y$. Hence EL and DK are parallel.

Angle CFM + angle CMF + angle $FCL = 180°$; by substitut'n $v + w - C = 120°$..(4).

If EL and FM are parallel, then angle MFC=angle ELC, and angle EMC =angle CEL, or $v = x + A + C - 60°$, and $w = y$. Substituting in (4), $A + x + y = 180°$ Since by (1) this relation is true, hence EL and FM are parallel.

107. Proposed by T. W. PALMER, A. M., Professor of Mathematics, University of Alabama.

Construct a triangle, given base, vertical angle and radius of inscribed circle.

I. Solution by H. C. WHITAKER, Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.

Denote the base by AB, the vertex by C, and the incenter by O. The angle AOB equals $90° + \frac{1}{2}C$ and hence one locus for O is the arc of a segment capable of containing this angle. Another locus is a parallel to the base the in-radius away. Hence the incircle can be constructed ; AC and BC are then drawn tangent to it.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Describe on the given base AB a circle the upper segment of which contains the given vertical angle. From the center O of this circle let fall the perpendicular on AB and produce it to D. At a distance from AB equal to the given radius of the inscribed circle draw MN parallel to AB. From D as a center with a radius equal to BD draw an arc cutting MN at E, connect E with D and extend DE until it

cuts the circumference at C, then will ABC be the required triangle. For, since $DE=BD$, 2 angle $EBD=180°-$angle $EDB=180°-A$.

$\therefore$ Angle $EBD=90°-\frac{1}{2}A$, but angle $EBA=$angle $EBD-$angle $ABD=$angle $EBD-\frac{1}{2}C=90°-\frac{1}{2}A-\frac{1}{2}C=\frac{1}{2}B$.

$\therefore$ BE is the bisector of B, and by construction, CD is the bisector of C.

$\therefore$ E is the center of the inscribed circle. Q. E. D.

Also solved by *G. B. M. ZERR, P. S. BERG, COOPER D. SCHMITT, F. H. POWE, F. W. HAMAWALT, ELMER SCHUYLER,* and the *PROPOSER.*

CALCULUS.

81. Proposed by J. OWEN MAHONEY, B. E., M. Sc., Instructor in Mathematics, Carthage High School, Carthage, Texas.

Solve : $y^2(d^2y/dx^2)+a(dy/dx)^2=bx$.

No solution of this problem has been received.

82. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Iowa.

A pole 60 feet high stands vertically in a river 20 feet deep. How many feet above the surface of the water must it break so that the top bending down would touch the bottom and the distance on the surface of water between the points where the parts of the pole enter the water would be a maximum?

I. Solution by C. HORNUNG, A. M., Professor of Mathematics, Heidelberg University, Tiffin, O., and GUY B. COLLIER, Union College, B. S. Course, Schenectady, N. Y.

Let $x=$the number of feet above the surface of the water the pole must break, and $y=$the number of feet between the parts of the pole on the surface of the water, which is to be a maximum.

By similar triangles we find $y=\dfrac{2x}{x+20}\sqrt{300+30x}$.

Simplifying and placing the first derivative equal to zero, we have a biquadratic in x whose roots are : 0, -20, 6.055, and -66.055. By substitution in the second derivative we find that 6.055 is the only one of these roots that renders y a maximum. Therefore $x=6.055$ is the required result.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let ABC represent the pole, BC being the part under water. Let D be the point where it breaks off, so that $DA=DE$. Let $AB=a$, $BC=b$, $BD=x$, $BF=y$; then $DA=DE=a-x$. $CE=\sqrt{[(a-x)^2-(b+x)^2]}=\sqrt{(a+b)}.\sqrt{(a-b-2x)}$ and $CE:y=b+x:x$, whence $y=\sqrt{a+b}.\dfrac{x}{b+x}\sqrt{a-b-2x}$.

$\therefore M=\dfrac{x^2}{(b+x)^2}(a-b-2x)$ is to be a maximum.

By differentiation we obtain after all the necessary and simple transformations the quadratic $x^2+3bx=(a-b)b$, whence $x=\frac{1}{2}[-3b+\sqrt{(5b^2+4ab)}]$.

For the numerical value $a=40$, $b=20$, we get $x=10(\sqrt{13}-3)=6.055$.

III. Solution by W. W. LANDIS, A. M., Professor of Mathematics and Astronomy, Dickinson College, Carlisle, Pa.; H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.; G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; O. S. WESTCOTT, A. M., Sc. D., Principal North Division High School, Chicago, Ill.; CHAS. E. MEYERS, Canton, Ohio; NELSON L. RORAY, Bridgetown, N. J.; CHAS. C. CROSS, Libertytown, Md.; ELMER SCHUYLER, High Bridge, N. J.; J. H. DRUMMOND, LL. D., Portland, Me.; COOPER D. SCHMITT, A. M. University of Tennessee, Knoxville, Tenn.; and the PROPOSER.

Let x=distance above the water, y=the required maximum.

Then $40-x$ is the hypotenuse and $\sqrt{[(40-x)^2-(20+x)^2]}=2\sqrt{(300-30x}$ =the base.

$\therefore y:x=2\sqrt{(300-30x)}:20+x.$

$\therefore y=\dfrac{2x\sqrt{(300-30x)}}{20+x}$=maximum.

Differentiating and reducing we get $(600-90x)(20+x)-2x(300-30x)=0$.

$\therefore x^2+60x=400.$

$\therefore x=6.0555128$ feet.

MECHANICS.

63. Proposed by A. H. BELL, Hillsboro, Ill.

From a horizontal support at a distance of 10 feet apart, a beam 5 feet long and 10 pounds weight is suspended by ropes attached to each end. The ropes are 3 and 5 feet respectively, in length. Required the angles made by the ropes and horizontal support. Also so the stress upon each rope.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa., and the PROPOSER.

Let $AB=10$, $BC=CD=5$, $AD=3$, $EB=x+y$, $AE=x-y$.

$$\cos AEB=\frac{(x+y)^2+(x-y)^2-100}{2(x+y)(x-y)}=\frac{(x+y-5)^2+(x-y-3)^2-25}{2(x+y-5)(x-y-3)}.$$

$\therefore (139-32x)y^2-(200-8x^2)y-79x^2+800x-1500=0$ ....(1).

$$DF=CN=\frac{100+(x-y)^2-(x+y)^2}{20(x-y)}(x-y-3)=\frac{100+(x+y)^2-(x-y)^2}{20(x+y)}(x+y-5).$$

$\therefore xy^3-xy^2-(x^3-4x^2+100)y+25x=0$....(2).

Let $139-32x=c$, $200-8x^2=2a$,

$79x^2-800x+1500=b$, $x^3-4x^2+100=d$.

$\therefore cy^2-2ay-b=0$.....(3),

and $xy^3-xy^2-dy+25x=0$.....(4).

y^2 of (3) in (4) $2axy^2+(bx-2ax-cd)y-bx+25cx=0$.....(5).

$(5)\times c-(3)\times 2ax$, $y=\dfrac{bcx-2abx-25c^2x}{bcx+4a^2x-2acx-c^2d}$ .....(6).

$(5)\times b+(3)\times(25cx-bx)$, $y=\dfrac{b^2x-50acx-bcd}{bcx-2abx-25c^2x}$ ....(7).

Now $b=25c-79a/4$ and $d=x^3+a$ in (6)=(7), etc.

$(316a^2x-558acx)^2+(16a^2x-4c^2x^3-4ac^2-87acx+100c^2x)\times$
$(6241a^2x+316acx^3+316a^2c-400c^2x^3-400ac^2-16600acx+10000c^2x)=0$ ....(8).

Expanding, we find a common factor c, then by substitution and reduction to the simplest form for the application of Horner's Method.

$$79x^{10}-1748x^9+12559x^8-2429.5x^7-478451.828125x^6$$
$$+2827762.5x^5-4080008.59375x^4-27582812.5x^3\cdot$$
$$+161863232.421875x^2-357007812.5x+301890625.=0....(9).$$

Horner's Method gives $x=7.95690209132$, (3) $y=0.564356664799$.
$x+y=8.521258756118$, $x-y=7.392545426520$.
$CE=3.521258756118$, $DE=4.392545426520$.
$\angle DAB=56^\circ\ 17'\ 54''$, $\angle ABC=46^\circ\ 11'\ 54''$, $\angle AEB=77^\circ\ 30'\ 12''$.
$\angle DCE=59^\circ\ 3'\ 32.5''$, $\angle CDE=43^\circ\ 26'\ 15.5''$.

Tension on $BC=\dfrac{10\cos DAB}{\sin(DAB+ABC)}=5.6863$ pounds.

Tension on $AD=\dfrac{10\cos ABC}{\sin(DAB+ABC)}=7.0896$ pounds.

[NOTE. By a mistake we published the incomplete solution of this problem in our last issue. Soon after receiving that solution, Dr. Zerr wrote us to the effect that a correct and complete solution would shortly follow. Mr. Bell, being ill at the time, was unable to send the complete solution at the time expected, so that by the time the Department was ready for the press, we forgot about the promised complete solution and sent in the incomplete solution for publication. We have not verified the above solution, and our readers must excuse us from that great task. We hold Dr. Zerr and Mr. Bell responsible for any errors contained in it.—ED. F.]

71. Proposed by the late B. F. BURLESON, Oneida Castle, N. Y.

Three men own a sphere of gold the density of which varies as the square of the distance from the center. If two segments be cut off each one inch from the center of the sphere it will be divided into three parts of equal value. Determine the diameter of the sphere.

II. Solution by R. E. GAINES, A. M., Professor of Mathematics, Richmond College, Richmond, Va.

The element $dydx$ whose ordinate is y will, when revolved about the axis of x generate an infinitesimal ring whose volume is $2\pi ydydx$, and whose distance from the center is $\sqrt{(x^2+y^2)}$. Therefore for the mass of the minor segment we have

$$2\int_0^1\int_0^{\sqrt{(a^2-x^2)}}2\pi y(x^2+y^2)dydx=\pi\int_0^1(a^4-x^4)dx=\pi(a^4-\tfrac{1}{5}).$$

Mass of sphere $=\pi\int_0^a (a^4-x^4)dx=\frac{4}{5}\pi a^5$.

$\therefore\ \frac{4}{5}\pi a^5=3\pi(a^4-\frac{1}{5}),\ 4a^5=15a^4-3$, which evidently has a root slightly less than 3.75.

In the solution given by Dr. Zerr in the November MONTHLY if the parts be added so as to give the mass of the sphere the result is not homogeneous in a and is therefore evidently wrong. In getting M the upper limit for θ should be $\cos^{-1}(1/r)$ and not $\cos^{-1}(1/a)$. For M_1 we must subtract $2M$ from the mass of the sphere.

75. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A particle P, is held in a bent tube by two forces directed towards two fixed points, H and S. Show that the equation of the tube is $PS.PH=k^2$, if the forces are μ/PS and μ/PH.

I. Solution by GEORGE R. DEANE, C. E., B. S., Professor of Mathematics, Missouri School of Mines, Rolla, Mo.

Put $PS=r_1$, $PH=r_2$. By the principle of virtual work, we have,

$$\frac{\mu}{r_1}\delta r_1+\frac{\mu}{r_2}\delta r_2=0.$$

Let $f(r_1, r_2)=0$ be the equation of the curve. Then

$$\frac{\partial f}{\partial r_1}\delta r_1+\frac{\partial f}{\partial r_2}\delta r_2=0.$$

Eliminating δr_1 and δr_2,

$$\frac{\frac{\partial f}{\partial r_1}}{\frac{\partial f}{\partial r_2}}=\frac{\frac{\mu}{r_1}}{\frac{\mu}{r_2}}.$$

Whence, $-\frac{dr_2}{dr_1}=\frac{r_2}{r_1}$, $r_1dr_2+r_2dr_1=0$, $r_1r_2=k^2$.

The general theorem of which this is a particular case, is given in Minchin's *Statics*, Vol. I., page 88.

II. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.

Let s=any arc of the tube, r, r'=the distances of the particle from the centers of force at any time t, m=the same absolute intensities of the forces, and β =the velocity of projection.

If S, S' be the radial forces attracting the particle, we will have

$$\frac{d^2s}{dt^2} = -S\frac{dr}{ds} - S'\frac{dr'}{ds} \ldots\ldots (1).$$

But $S = m/r$, $S' = m/r'$, and (1) becomes

$$\frac{d^2s}{dt^2} = -\frac{m}{r}\frac{dr}{ds} - \frac{m}{r'}\frac{dr'}{ds} \ldots\ldots (2).$$

Multiply by $2(ds/dt)$ and integrate ; then

$$\frac{ds^2}{dt^2} = -m\log r^2 - m\log r'^2 + C \ldots\ldots (3).$$

When $r = a$, $r' = a$, $\frac{ds}{dt} = \beta$; $\therefore\ C = \beta^2 - m^2\log\frac{1}{a^4}$, and (3) is

$$\beta^2 = m^2\log\frac{1}{r^2r'^2} + \beta^2 - m^2\log\frac{1}{a^4} \quad \ldots\ (4),$$

or, $rr' = a^2 \ldots\ldots (5)$, a lemniscate.

[Other solutions of this problem will appear in the next issue.]

DIOPHANTINE ANALYSIS.

73. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find integral values for x and y in $\begin{pmatrix} 2x^2 - y^2 = \square \\ 2y^2 - x^2 = \square \end{pmatrix}$.

I. Solution by the PROPOSER.

$$2x^2 - y^2 = \square = a^2 \ldots\ldots (1),\ 2y^2 - x = \square = (2).$$

From (1), $y^2 = 2x^2 - a^2$. Substituting this in (2), we have $3x^2 - 2a^2 = b^2$. Whence $x = \frac{1}{3}\sqrt{[3(2a^2 + b^2)]}$, and $y = \frac{1}{3}\sqrt{[3(a^2 + 2b^2)]}$.

As far as I know, the only method of rationalizing both radicals is to put $a = b$. Then $x = y = a = b$.

Accordingly, no *different* integral values can be found for x and y.

This problem is the key to Problem 62, "To find four squares in arithmetical progression."

The roots of the four squares would then be, respectively,

$$a,\quad \tfrac{1}{3}\sqrt{[3(2a^2 + b^2)]},\quad \tfrac{1}{3}\sqrt{[3(a^2 + 2b^2)]},\quad b.$$

The common difference of the squares is $\frac{1}{3}(b^2 - a^2)$.

According to the above solution, the roots of the four squares could not *all* be rational integers ; *one* of them, at least, must be a *surd*. It is evident, however, that an infinite number of sets of four squares can be found in which *two* of the roots are rational integers.

Put $a=1$ and $b=2$. Then the roots of the four squares are 1, $\sqrt{2}$, $\sqrt{3}$, 2.

Put $a=1$ and $b=5$. Then the roots are 1, 3, $\sqrt{17}$, 5.

A similar proof was received from *CHARLES C. CROSS*.

II. Solution by **A. H. BELL, Hillsboro, Ill.**

Take $2y^2-x^2=\square$, or $x^2-2y^2=-\square=-1=-4$, etc.

In $x^2-2y^2=-1$.....(3), the integral values for x and y are the alternate convergent fractions for the $\sqrt{2}$=to

$$x/y=1/1,\ 7/5,\ 41/29,\ \text{etc}.....(4).$$

For the next, $x^2-2y^2=-4$. $(4)\times\sqrt{4}$,

$$\frac{x}{y}=\frac{1\times 2}{1\times 2},\quad \frac{7\times 2}{5\times 2},\quad \frac{41\times 2}{29\times 2},\quad \text{etc.}$$

Consequently the interchangeable values of x and y must be found in the first fraction and no other.

III. Solution by **JOSIAH H. DRUMMOND, LL. D., Portland, Me.**

$2x^2-y^2=\square$ .....(1), and $2y^2-x^2=\square$ .....(2).

Take $x=my$ and $2m^2-1=\square$ .....(3), and $2-m^2=\square$ .....(4).

Then $m<\sqrt{2}$ and $m>\frac{1}{2}\sqrt{2}$.

It is manifest that both (3) and (4) are rational when $m=1$, which is $<\sqrt{2}$ and $>\frac{1}{2}\sqrt{2}$.

Then in (3) take $m=n+1$, and we have

$$2n^2+4n+1=\square=(\text{say})(qn-1)^2, \text{ whence}$$

$$n=\frac{2(q+2)}{q^2-2} \text{ and } m=n+1=\frac{(q+1)^2+1}{q^2-2}.$$

Substituting this value of m in (4) and reducing by the usual methods, we find $q=0$ and $m=\pm 1$.

Hence $x=\pm y$ and the integral values are any equal numbers, positive or negative, or one positive and the other negative.

MISCELLANEOUS.

65. Proposed by **J. M. COLAW, A. M., Monterey, Va.**

Three circles, radii in ratio 1, 3, 5, are tangent externally and enclose one acre; what are the radii ?

Solution by A. H. BELL, Hillsboro, Ill.; J. SCHEFFER, A. M., Hagerstown, Md.; FREMONT CRANE, Sand Coulee, Mont., and COOPER D. SCHMITT, A. M. University of Tennessee, Knoxville, Tenn.

Let the radii be x, $3x$, and $5x$, respectively.

The sides of the triangle formed by joining the centers are $4x$, $6x$, and $8x$, and let the angles opposite each side, respectively, be A, B, and C; and then are found from the sides of the initial triangle 2, 3, and 4,

$$\cos A = \frac{9+16-4}{24}. \quad \therefore A = 28^\circ\ 57'\ 18''\ \text{arc} = .5053601,$$

$$\cos B = \frac{4+16-9}{16}. \quad \therefore B = 46^\circ\ 34'\ 3''\ \text{arc} = .8127562,$$

$$\cos C = \frac{4+9-16}{12} = -\tfrac{1}{4}. \quad \therefore C = 104^\circ\ 28'\ 39''\ \text{arc} = 1.8234763.$$

Now the triangle equals the three seetors = one acre = (10 square chains).

Then $$\frac{25x^2A}{2} + \frac{9x^2B}{2} + \frac{x^2C}{2} + 10 = 24x^2 \sin A \quad \ldots\ldots\ \ldots\ldots\ \ldots(1).$$

$$\therefore x = \sqrt{\frac{20}{48 \sin A - (25A + 9B + C)}} \quad \ldots\ldots\ \ldots\ldots\ \ldots\ldots\ \ldots\ldots\ \ldots\ldots\ (2).$$

$1 - \cos^2 A = \sin^2 A$ and $\sin A = \frac{1}{8}\sqrt{15} = 0.4841229$.

$x = 3.694$ chains, and the radii are 3.694, 11.082, and 18.47 chains.

Also solved by *G. B. M. ZERR*, and the *PROPOSER*.

Solved by Elmer Schuyler, with results, 9.985 rods, 29.955 rods, and 49.925 rods; by Alois F. Kovarik with 19.903 rods, 57.709 rods, and 99.515 rods; and by Josiah H. Drummond with approximate results. The methods were all correct but there were some errors of calculation. Later a solution was received from Walter H. Drane, with results, 211.329+ feet, 633.987+ feet, and 1056.645+ feet.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

108. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A man who feels his death approach bequeathes to his young wife one-third of his fortune, and the remaining two-thirds to his son, if such should be born; but one-half of it to the widow and the other half to his daughter, if such should be born. After his death twins are born, a son and a daughter. How should the fortune be divided amongst the three?

109. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Why do fences and telegraph poles appear to move rapidly in an opposite direction to one traveling in a railway car? [From Moore's *Grammar School Arithmetic*, page 150.]

*** Solutions of these problems should be sent to B. F. Finkel not later than April 10.

ALGEBRA.

96. Proposed by F. M. PRIEST, Mona House, St. Louis, Mo.

How many different numbers may be written with the nine digits and zero, using them singly and in groups of from two to ten digits each, and using no figure but once in each group? How many more nnmbers may be written by repeating the digits and zero at pleasure in each group?

97. Proposed by F. M. SHIELDS, Coopwood, Miss.

A farmer had 2080 pounds of grain at the depot, and gave a wagoner .75 cents per 100 pounds to haul it, paying him in the *same* grain at the following prices, viz.: 3-10 of the hauling bill was paid in corn at .58 cents per bushel of 56 pounds. 3-5 was paid in wheat at 1.55 cents per pushel of 60 pounds, and the balance of the bill was paid in oats at .36 cents per bushel of 32 pounds, the wagoner not charging for hauling his own grain. The load being delivered, how many bushels of each kind of grain did the wagoner get, and how many bushels of each kind did the farmer have left after paying the wagoner?

*** Solutions of these problems should be sent to J. M. Colaw not later than April 10.

GEOMETRY.

116. Proposed by P. S. BERG, A. M., Superintendent of Schools, Larimore, N. D.

Inscribe by rule and compass a regular heptadecagon.

117. Proposed by GUY B. COLLIER, Schenectady, N. Y.

If (x', y') and (x'', y'') are the extremities of a pair of conjugate diameters whose eccentric angles are φ' and φ, show that $\varphi' + \varphi = 90°$; given $(x', y') = (a\sec\varphi', b\tan\varphi)$. [From Nichols' *Analytical Geometry*.]

*** Solutions of these problems should be sent to B. F. Finkel not later than April 10.

MECHANICS.

84. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Two weights P and Q are fastened by a weightless string that is strung over a single movable pulley. P is greater than Q. The weight of the pulley is $2R$. Find the tension of the string, (1) when the friction of the string on the pulley is neglected, (2) when it is considered.

85. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.

A circular tube of radius a revolves uniformly about a vertical diameter with angular velocity $\sqrt{\frac{ng}{a}}$, and a particle is projected from its lowest point with such velocity that it can just reach the highest point; prove that the time of describing the first quadrant is $\sqrt{\frac{a}{(n+1)g}}\log(\sqrt{n+2}+\sqrt{n+1})$.

86. Prize Problem ; $2.50 for the best solution.

Two spheres of equal size are in motion on a smooth horizontal plane, and, on meeting, their plane of contact coincides with the plane of the meridian. The sphere on the west side is perfectly elastic and weighs 4½ pounds, while previous to the impact it was moving N. 30° E. with a velocity of 15 feet per second. The sphere on the east side is perfectly plastic and weighs 6½ pounds, while previous to the impact it was moving N. 45° W. with a velocity of 10 feet per second. Determine the motions of the spheres after the impact.

*** Solutions of these problems should be sent to B. F. Finkel not later than April 10.

EDITORIALS.

Our valued contributor, J. M. Bandy, A. M., Ph. D., is now engaged as chief engineer of the Cape Fear and Northern Railroad.

The mathematical text-books formerly published by Leach, Shewell & Sanborn, of Boston, have been purchased by D. C. Heath & Co. Among the most valuable of these text-books are Osborne's *Differential and Integral Calculus*, Nichols' *Analytic Geometry*, and Fine's *Number System of Algebra*.

Through the kindness of a subscriber, who desires that his name should not be mentioned, we are able to offer a prize for the best solution of Problem 86, Mechanics, published in this issue, furnished by a person under the age of twenty-one years. All solutions must be forwarded to B. F. Finkel on or before May 1, 1899. The object of the donor in giving this prize is to create an interest in mathematics among teachers and young people. The first, second, and third solutions in order of neatness and accuracy will be published.

BOOKS AND PERIODICALS.

Plane and Solid Geometry. By James Howard Gore, Ph. D., Professor of Mathematics in Columbian University, Author of Elements of Geodesy, History of Geodesy, Bibliography of Geodesy, etc., etc. 8vo. Cloth, 210 pages. Price, $1.00. New York : Longmans, Green & Co.

The object of this work seems to be to bring the study of Geometry within the minimum time requisite to gain a fair knowledge of it, in order that a proportionate amount of time may be given to other subjects. The author holds that since other sciences, and even language and philosophy, claim disciplinary merit equal to that possessed by mathematics, the time has come when we can afford to hearken to the demands of the utilitarians and give up those refinements in mathematics which have been retained for the mental discipline they bring about, but which are wholly lacking in practical application.

From this point of view, the author has eliminated from this work all propositions that are not of practical value or needed in the demonstrations of such propositions, and thus about half the matter usually found in geometries is omitted. The author makes one very sound statement in his preface, viz., "While symbols and equational statements have the advantage of brevity and convey information to the mind through its most receptive channel,—eye,—still they discourage the use of language, and hence fail to develop by example and precept the employment of accurate and precise forms of expression." While this statement is true, yet the very concise and symbolic statements made in most of our recent texts on geometry offer no serious objection, if the teacher is careful to see that the student does not fall into a careless and slovenly mode of expression.

We do not agree with the author's utilitarian view except in cases where the object of the study is utilitarian in purpose. In college courses, no subject should be studied simply with a view of its practical value in after life. If such were the aim, many subjects, as for example, Latin, Greek, Chemistry, many branches of Mathematics, etc., might be omitted from the college course. But since the object of the course is disciplinary rather than utilitarian, mathematics must and always will receive its proportionate time, and hence many of the modern refinements of geometry should remain that the study of the subject may result in the highest mental development. B. F. F.

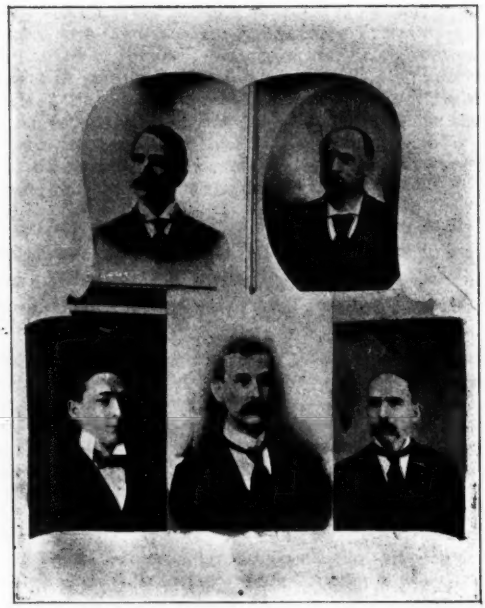
Problèms de Geometrie Élémentaire Groupés D'aprés les Méthodes à Employer pour leur résolutions. Par Ivan Alexandroff, Professeur de Mathematiques au Lycée de Tambov (Russie), Traduit du russie, sur la sixième èdition, Par D. Aitoff. 8vo. Paper Cover, 156 pages. Paris, France: Librairie Scientifique A. Hermann.

In this work are discussed problems on the construction of geometric figures, methods of geometric loci, method of similitude, method of contrary problems, method of symmetry, method of translation, method of rotation about an axis, method of rotation about a point, method of inversion, and application of algebra to geometry. Under each of these subjects are given demonstrations of type-propositions, that is, propositions setting forth the chief principles employed in the demonstrations of the propositions under the subject considered. For example, under *Problems on the Similitude of Figures*, the following problem comes first: Construct a triangle, having given an angle, the ratio of the sides which contain this angle and the radius of the inscribed circle. This problem is solved and its solution illustrated with a figure. Then comes this problem: Construct a triangle, having given an angle α, the length s of the bisector of this angle and the ratio m/n of the segments determined on the side opposite to α by the altitude from the vertex of this angle. This problem is also solved and the solution illustrated with a figure. Then comes this problem: Having given an angle ABC and a point M on the interior of this angle, find on the side BC a point X equidistant from the side AB and from the point M. Six other problems whose solutions are given follow. These are followed by a large number of exercises. This work in the hands of a good teacher will add interest in his work and profit to his class. B. F. F.

Sur la Résolution de L' Equation du Troisieme Degré. By Dr. Alexander Macfarlane.

In this paper Dr. Macfarlane has applied the hyperbolic calculus to the solution of the cubic equation, and the Method of Cardan and Trigonometric Method are united in a general method applicable to all cases. B. F. F.

Calcul de Généralization. Par G. Oltramare Doyen de la Faculté des Sciences de L' Université de Genéve. Paper Cover, 192 pages. Paris, France: Librairie Scientifique A. Hermann.



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ON SYMMETRIC FUNCTIONS.

By DR. E. D. ROE, Jr., Associate Professor of Mathematics in Oberlin College.

[Continued from February Number.]

c. With the first four equations, we have disposed of 16 of the 20 functions. Only four of them needed calculating. They were :

$$b_0^3\Sigma\beta_1^3\beta_2^3,\ b_0^2\Sigma\beta_1^2\beta_2^2,\ b_0^3\Sigma\beta_1^3,\ b_0^2\Sigma\beta_1^2.$$

The remaining 12 were the six functions already mentioned under *a*, and the following six :

$$|\,0^2 1\,| = b_0^3\Sigma\beta_1^3\beta_2^3\beta_3^2 = b_0^2(\beta_1\beta_2\beta_3)^2 b_0\Sigma\beta_1\beta_2,$$

$$|\,0^2 2\,| = b_0^3\Sigma\beta_1^3\beta_2^3\beta_3 = b_0(\beta_1\beta_2\beta_3)b_0^2\Sigma\beta_1^2\beta_2^2,$$

$$|\,01^2\,| = b_0^3\Sigma\beta_1^3\beta_2^3\beta_3^2 = b_0(\beta_1\beta_2\beta_3)^2 b_0\Sigma\beta_1,$$

$$|\,1^2 2\,| = b_0^3\Sigma\beta_1^2\beta_2^2\beta_3 = b_0 b_0(\beta_1\beta_2\beta_3)b_0\Sigma\beta_1\beta_2,$$

$$|\,02^2\,| = b_0^3\Sigma\beta_1^3\beta_1\beta_2 = b_0(\beta_1\beta_2\beta_3)b_0^2\Sigma\beta_1^2,$$

$$|\,12^2\,| = b_0^3\Sigma\beta_1^2\beta_2\beta_3 = b_0 b_0(\beta_1\beta_2\beta_3)b_0\Sigma\beta_1.$$

d. The remaining four functions may be obtained from four of the last six equations. The functions are :

$$|\,012\,| = b_0^3 \Sigma\beta_1^3\beta_2^2\beta_3 = b_0(\beta_1\beta_2\beta_3)b_0^2\Sigma\beta_1^2\beta_2,$$

$$|\,013\,| = b_0^3\Sigma\beta_1^3\beta_2^2,$$

$$|\,023\,| = b_0^3\Sigma\beta_1^3\beta_2,$$

$$|\,123\,| = b_0^3\Sigma\beta_1^2\beta_2.$$

On account of the decomposition of the first into a known function and one of the other three, only three of these four require actual calculation.

3. Continuation of the Method.

This method may be continued for higher resultants, what was said under (3) regarding the analysis of the operator δ of Aronhold into three others being capable of the easiest extension to the general case in that in the statements there made one must substitute the numbers 0, 1, 2, ...n, and the literal factors b_0, b_1, ...b_n instead of 0, 1, 2, 3, and b_0, b_1, b_2, b_3, respectively, and then these operators must be applied to all the $\frac{(n+1)(n+2).....(2n-1)}{(n-1)!}$ stroked forms of $(n+1)$ elements to $(n-1)$ dimensions, thus obtaining $\frac{(n+1)(n+2)....(2n-1)}{(n-1)!}$ identical equations from which the $\frac{(n+1)(n+2)....2n}{n!}$ symmetric functions in connection with the $2n$ which may be already assumed as known, can be found. We have thus at the same time a method for elimination by means of symmetric functions, and a method for computing the values of the symmetric functions, giving the function as a whole. The functions for intermediate forms where m and n are unequal ($m>n$) are contained in those forms where $n=m$.

II. ISOLATION OF TERMS OF A SYMMETRIC FUNCTION.

A. Preliminary Statements Concerning the Resultant Theory.

1. The Concept of Normal and Reducible Forms.

(1). *Definitions.*

In the thesis of the writer entitled "Die Entwickelung der Sylvester'schen Determinante nach Normal-Formen," (B. G. Teubner, Leipzig, 1898) the forms

$$P_{m,n} = a_{\kappa_1}a_{\kappa_2}....a_{\kappa_m}b_{\lambda_1}b_{\lambda_2}...b_{\lambda_m} = (a_{r_1})^{p_1}(a_{r_2})^{p_2}....(a_{r_\mu})^{p_\mu}(b_{s_1})^{q_1}(b_{s_2})^{q_2}....(b_{s_\nu})^{q_\nu},$$

which occur in the resultant $R_{m,n}$ of

$$f = a_0x^m - a_1x^{m-1} + ...(-1)^m a_m \text{ and } \phi = b_0x^n + b_1x^{n-1} + ... + b_n$$

are divided into such as have the four factors a_0, b_0, a_m, b_n, and those which do